

# Regularity of the diffusion-dispersion tensor and error analysis of Galerkin FEMs for a porous media flow

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## Abstract

We study Galerkin finite element methods for an incompressible miscible flow in porous media with the commonly-used Bear–Scheidegger diffusion-dispersion tensor  $D(\mathbf{u}) = \Phi d_m I + |\mathbf{u}|(\alpha_T I + (\alpha_L - \alpha_T) \frac{\mathbf{u} \otimes \mathbf{u}}{|\mathbf{u}|^2})$ . The traditional approach to optimal  $L^\infty((0, T); L^2)$  error estimates is based on an elliptic Ritz projection, which usually requires the regularity of  $\nabla_x \partial_t D(\mathbf{u}(x, t)) \in L^p(\Omega_T)$ . However, the Bear–Scheidegger diffusion-dispersion tensor may not satisfy the regularity condition even for a smooth velocity field  $\mathbf{u}$ . A new approach is presented in this paper, in terms of a parabolic projection, which only requires the Lipschitz continuity of  $D(\mathbf{u})$ . With the new approach, we establish optimal  $L^p$  error estimates and an almost optimal  $L^\infty$  error estimate.

## 1 Introduction

The flow of incompressible miscible fluids in porous media was extensively investigated in the last several decades [10, 11, 29] due to its wide applications in engineering, such as reservoir simulations and exploration of underground water, oil and gas. The problem is governed by the following equations:

$$\Phi \frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u}) \nabla c) + \mathbf{u} \cdot \nabla c = \hat{c} q_i - c q_p, \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = q_i - q_p, \quad (1.2)$$

$$\mathbf{u} = -\frac{k(x)}{\mu(c)} \nabla P, \quad (1.3)$$

where  $P$  and  $\mathbf{u}$  are the pressure and velocity of the mixture of two fluids, respectively,  $c$  is the concentration of one fluid,  $k$  is the permeability of the porous medium,  $\mu(c)$  is the concentration-dependent viscosity of the fluid mixture,  $\Phi$  is the porosity of the medium,  $q_i$  and  $q_p$  are the given injection and production sources, respectively,  $\hat{c}$  is the injected concentration, and  $D(\mathbf{u}) = [D_{ij}(\mathbf{u})]_{d \times d}$  denotes the diffusion-dispersion tensor. We assume that the system is defined in a bounded and smooth domain  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , for  $t \in [0, T]$ , subject to the boundary and initial conditions:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= 0, \quad D(\mathbf{u}) \nabla c \cdot \mathbf{n} = 0 \quad \text{for } x \in \partial\Omega, \quad t \in [0, T], \\ c(x, 0) &= c_0(x) \quad \text{for } x \in \Omega, \end{aligned} \quad (1.4)$$

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under the compatibility condition

$$\int_{\Omega} q_i \, dx = \int_{\Omega} q_p \, dx. \quad (1.5)$$

Numerical analysis for the above system was done for a variety of numerical methods, such as the Galerkin-Galerkin finite element methods (FEMs), the Galerkin-mixed FEMs, the method of characteristics type and discontinuous Galerkin methods [9, 12, 14–16, 25, 26, 32, 36, 37, 39]. Mathematical analysis, existence and uniqueness of solutions of the system, was investigated in [17]. In the above system, the diffusion-dispersion tensor could be different in different applications [22, 38]. A popular one used in reservoir simulations and exploration of underground water, oil and gas is [4, 5]

$$D(\mathbf{u}) = \Phi d_m I + F(Pe, r) |\mathbf{u}| \left( \alpha_T I + (\alpha_L - \alpha_T) \frac{\mathbf{u} \otimes \mathbf{u}}{|\mathbf{u}|^2} \right) \quad (1.6)$$

where  $d_m > 0$  denotes the molecular diffusion,  $\alpha_L$  and  $\alpha_T$  denote the constant longitudinal and transversal dispersivities of the isotropic porous medium, respectively, and  $F(Pe, r)$  is a function of the local molecular Peclet number and the ratio of length characterizer of the porous medium in general. The commonly-used formulation of the function is the Bear–Scheidegger dispersion model [22, 35], in which

$$F(Pe, r) = 1. \quad (1.7)$$

The Bear–Scheidegger dispersion model has been widely used for numerical simulations and analysis. An important issue in numerical analysis is the regularity of the diffusion-dispersion tensor. It was shown in [31, 36] that the Bear–Scheidegger diffusion-dispersion tensor  $D(\mathbf{u})$  is Lipschitz continuous in  $\mathbf{u}$ . However, we notice that its second-order derivatives may not be bounded around  $|\mathbf{u}| = 0$ . For example, in the case  $\alpha_L = \alpha_T$ , the smooth velocity field

$$\mathbf{u} = (x_1 - x_2 - t) \mathbf{e}_1$$

satisfies that  $\mathbf{u} \in W^{2,\infty}(\overline{\Omega} \times [0, T])$  and  $D(\mathbf{u}) \in W^{1,\infty}(\overline{\Omega} \times [0, T])$ , while

$$\nabla_x \partial_t D(\mathbf{u}(x, t)) \notin L^p(\Omega_T) \quad \text{for any } p \geq 1.$$

Such a weak regularity of the Bear–Scheidegger diffusion-dispersion tensor may not affect numerical simulations of some practical problems, while it could be serious in numerical analysis, particularly for optimal error estimates of FEMs.

A traditional approach to establish the optimal  $L^\infty((0, T); L^2)$ -norm error estimate is to introduce an elliptic Ritz projection  $\mathbf{R}_h(t) : H^1(\Omega) \rightarrow S_h^r$  [13, 41] defined by

$$\left( D(\mathbf{u}(\cdot, t)) \nabla(\phi - \mathbf{R}_h \phi), \nabla \varphi_h \right) = 0, \quad \text{for all } \phi \in H^1(\Omega) \text{ and } \varphi_h \in S_h^r, \quad (1.8)$$

where  $S_h^r$  denotes the finite element space. Many previous works on optimal  $L^\infty((0, T); L^2)$  error estimates of Galerkin or mixed FEMs for the nonlinear parabolic system (1.1)–(1.3) were based on this approach, which required the a priori estimate

$$\|\partial_t(c - \mathbf{R}_h c)\|_{L^2(\Omega \times (0, T))} \leq Ch^{r+1}. \quad (1.9)$$

The above estimate was established in [41], under the regularity assumption

$$\|\nabla_x \partial_t D(\mathbf{u}(x, t))\|_{L^\infty(\Omega \times (0, T))} \leq C \quad (1.10)$$

for a general nonlinear parabolic equation. Since the Bear–Scheidegger dispersion model does not satisfy the regularity condition (1.10), optimal  $L^\infty((0, T); L^2)$  error estimates of Galerkin–Galerkin methods, Galerkin–mixed methods and many other numerical methods for this model have not been well done, while all the proofs in those previous works are valid only for some other dispersion models [5]. In addition, some special cases were studied by several authors with some other techniques. An optimal-order  $L^\infty((0, T); L^2)$  error estimate of a nonsymmetric DG method for a linear diffusion equation was obtained in [40] without the use of the elliptic projection. The analysis in [40] was limited in a bilinear (or trilinear) FE approximation on a uniform mesh, for which the superconvergence of the corresponding interpolation can be utilized. More recently, a combined method with a DG time discretization and (mixed) FE approximations in the spatial direction was proposed in [31] for the miscible displacement equations (1.1)–(1.3) with the Bear–Scheidegger dispersion model and low regularity of the solution. The convergence of numerical solution to the exact solution of the equations was proved.

In this paper, we study the commonly-used Bear–Scheidegger dispersion model by Galerkin FEMs and establish an optimal  $L^p$  error estimate, as well as an almost optimal  $L^\infty$  error estimate. Here we introduce a new approach, in terms of a parabolic projection. Our analysis relies on the fact that for the Bear–Scheidegger model,  $D(\mathbf{u}) \in W^{1,\infty}(\overline{\Omega} \times [0, T])$ . The key to our analysis is an  $L^p$ -norm stability estimate of the finite element solution for the linear parabolic equation

$$\begin{cases} \partial_t \phi - \nabla \cdot (A \nabla \phi) + \phi = f - \nabla \cdot \mathbf{g} & \text{in } \Omega, \\ A \nabla \phi \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n} & \text{on } \partial\Omega, \\ \phi(x, 0) = \phi_0(x) & \text{for } x \in \Omega, \end{cases} \quad (1.11)$$

whose finite element solution  $\{\phi_h(t)\}_{t>0}$  is defined by

$$\begin{cases} (\partial_t \phi_h, v_h) + (A \nabla \phi_h, \nabla v_h) + (\phi_h, v_h) = (f, v_h) + (\mathbf{g}, \nabla v_h), & v_h \in S_h^r, \\ \phi_h(\cdot, 0) = \phi_h^0, \end{cases} \quad (1.12)$$

where  $\phi_h^0$  is a certain approximation to the initial data  $\phi_0$ . The finite element solution  $\phi_h$  can be viewed as a parabolic projection of  $\phi$  onto the finite element space. In fact, many efforts have been devoted to the stability estimate of the parabolic projection:

$$\|\phi_h\|_{L^\infty(\Omega \times (0, T))} \leq \|\phi_h^0\|_{L^\infty(\Omega)} + C l_h \|\phi\|_{L^\infty(\Omega \times (0, T))},$$

e.g., see [7, 18, 19, 24, 28, 33]. These estimates were obtained only for autonomous parabolic equations whose coefficients are smooth enough, i.e.,  $A = A(x) \in C^{2+\alpha}(\overline{\Omega})$ . In a recent work [27], the first author established the  $L^p$ -norm ( $1 < p \leq \infty$ ) stability estimates for parabolic equations with Lipschitz continuous coefficients  $A = A(x) \in W^{1,\infty}(\Omega)$ . In this paper, we shall extend these  $L^p$ -norm estimates to nonautonomous parabolic equations with the coefficient  $A(x, t) \in L^\infty((0, T); W^{1,\infty}(\Omega)) \cap C(\overline{\Omega} \times [0, T])$  and then, apply these estimates to the nonlinear equations for incompressible miscible flows in porous media with the Bear–Scheidegger dispersion model to obtain optimal  $L^p$  and almost optimal  $L^\infty$  error estimates of Galerkin FEMs. Our theoretical analysis provides a new fundamental tool in establishing optimal error estimates of Galerkin FEMs for nonlinear parabolic equations with Lipschitz continuous coefficients.

The rest part of this paper is organized as follows. In Section 2, we introduce some notations and present our main results. In Section 3, we establish an optimal  $L^p$  error estimate and an almost optimal  $L^\infty$  error estimate of Galerkin FEMs for the equations of the incompressible

miscible flow, based on the  $L^p$ -norm stability estimate of the finite element solution for the linear parabolic equation (1.11). The proof of the  $L^p$ -norm stability estimate will be given in Section 4. Two numerical examples are given in Section 5 to confirm our theoretical analysis.

## 2 Notations and main results

Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^d$ ,  $d \geq 2$ . For any nonnegative integer  $k$ , we let  $W^{k,p} = W^{k,p}(\Omega)$ ,  $1 \leq p \leq \infty$ , denote the usual Sobolev spaces [1]. For any negative integer  $k$ , we denote by  $W^{-k,p}$  the dual space of  $W^{k,p'}$ . We adopt the convention  $L^p = W^{0,p}$  and  $H^k = W^{k,2}$  for any integer  $k$ . Let  $\Omega_T := \Omega \times (0, T)$  and for any function  $f$  defined on  $\Omega_T$  we use the abbreviation  $f(t)$  to denote the function  $f(\cdot, t)$  defined on the domain  $\Omega$  for the given  $t \in [0, T]$ .

Let the domain  $\Omega$  be partitioned into quasi-uniform triangles (or tetrahedras)  $T_j$ ,  $j = 1, \dots, J$ , which fit the boundary exactly. For the given positive integer  $r$  we define the finite element space subject to the triangulation by

$$S_h^r = \{\phi_h \in C(\overline{\Omega}) : \phi_h \text{ is a polynomial of degree } r \text{ on each triangle } T_j\}.$$

Suppose that  $A(\cdot, t) \in W^{1,\infty}$  and  $K^{-1}|\xi|^2 \leq A_{ij}(x, t)\xi_i\xi_j \leq K|\xi|^2$  for any  $\xi \in \mathbb{R}^d$  and  $x \in \Omega$ , where  $K$  is some positive constant. Then we define the operator  $\mathbf{A}(t) : H^1 \rightarrow H^{-1}$  by

$$(\mathbf{A}(t)\phi, \varphi) = (A(t)\nabla\phi, \nabla\varphi) + (\phi, \varphi) \quad \text{for any } \phi, \varphi \in H^1,$$

and let  $\mathbf{R}_h(t) : H^1 \rightarrow S_h^r$  be the Ritz projection operator associated with the elliptic operator  $\mathbf{A}(t)$ , i.e.,

$$(A(t)\nabla(\phi - \mathbf{R}_h(t)\phi), \nabla\varphi_h) + (\phi - \mathbf{R}_h(t)\phi, \varphi_h) = 0 \quad \text{for any } \phi \in H^1 \text{ and } \varphi_h \in S_h^r.$$

Let  $\mathbf{P}_h$  be the  $L^2$  projection operator onto the finite element space defined by

$$(\phi - \mathbf{P}_h\phi, \varphi_h) = 0 \quad \text{for any } \phi \in L^2 \text{ and } \varphi_h \in S_h^r,$$

and define the operator  $\mathbf{A}_h(t) : H^1 \rightarrow S_h^r$  by

$$(\mathbf{A}_h(t)\phi, \varphi_h) = (A(t)\nabla\phi, \nabla\varphi_h) + (\phi, \varphi_h) \quad \text{for any } \phi \in H^1 \text{ and } \varphi_h \in S_h^r$$

so that  $\mathbf{A}_h(t)\mathbf{R}_h(t)\phi = \mathbf{A}_h(t)\phi$  for  $\phi \in H^1$ . Moreover, we define the operator  $\overline{\nabla} \cdot : (L^2)^d \rightarrow H^{-1}$  by

$$(\overline{\nabla} \cdot \mathbf{w}, v) = -(\mathbf{w}, \nabla v) \quad \text{for } \mathbf{w} \in (L^2)^d \text{ and } v \in H^1.$$

and define the operator  $\overline{\nabla}_h \cdot : (L^2)^d \rightarrow S_h^r$  by

$$(\overline{\nabla}_h \cdot \mathbf{w}, v_h) = -(\mathbf{w}, \nabla v_h) \quad \text{for } \mathbf{w} \in (L^2)^d \text{ and } v_h \in S_h^r.$$

With the above definitions, the  $L^2$  projection operator  $\mathbf{P}_h$  and the Ritz projection operator  $\mathbf{R}_h(t)$  onto the finite element space are well defined and satisfy [6, 21, 30, 34]

$$\|\varphi - \mathbf{P}_h\varphi\|_{W^{l_0,q}} \leq Ch^{m-l_0}\|\varphi\|_{W^{m,q}}, \quad \forall \varphi \in W^{m,q}, \quad (2.1)$$

$$\|\mathbf{P}_h\varphi - \mathbf{R}_h(t)\varphi\|_{L^q} + h\|\mathbf{P}_h\varphi - \mathbf{R}_h(t)\varphi\|_{W^{1,q}} \leq Ch^{l+1}\|\varphi\|_{W^{l+1,q}}, \quad \forall \varphi \in W^{l+1,q}, \quad (2.2)$$

for  $l_0 = 0, 1$ ,  $l_0 \leq m \leq r + 1$  and any integer  $0 \leq l \leq r$ . The last two inequalities immediately imply [6] that the finite element solution  $u_h$  (with  $\int_{\Omega} u_h \, dx = 0$ ) of the equation

$$(A(t)\nabla u_h, \nabla v_h) = (\mathbf{f}, \nabla v_h), \quad \forall v_h \in S_h^r,$$

satisfies the  $W^{1,q}$  estimate

$$\|u_h\|_{W^{1,q}} \leq C\|\mathbf{f}\|_{L^q}. \quad (2.3)$$

Our first result of this paper is the following theorem concerning an  $L^p$  stability estimate and the maximal  $L^p$  regularity of the finite element solution for nonautonomous parabolic equations with nonsmooth coefficients.

**Theorem 2.1 (L<sup>p</sup> stability and maximal L<sup>p</sup> regularity)**

If the symmetric matrix  $A = (A_{ij})_{d \times d}$  satisfies that  $A_{ij}(x, t) \in L^\infty((0, T); W^{1,\infty}) \cap C(\overline{\Omega}_T)$  and

$$K^{-1}|\xi|^2 \leq \sum_{i,j=1}^d A_{ij}(x, t)\xi_i\xi_j \leq K|\xi|^2, \quad \forall \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \quad \text{and} \quad (x, t) \in \Omega_T$$

for some positive constant  $K$ , then the solutions of (1.11) and (1.12) satisfy the  $L^p$  stability estimate

$$\|\mathbf{P}_h\phi - \phi_h\|_{L^p((0,T);L^q)} \leq C_{p,q}\|\mathbf{P}_h\phi^0 - \phi_h^0\|_{L^q} + C_{p,q}\|\mathbf{P}_h\phi - \mathbf{R}_h\phi\|_{L^p((0,T);L^q)} \quad (2.4)$$

and the maximal  $L^p$  regularity estimate

$$\|\partial_t\phi_h\|_{L^p((0,T);W^{-1,q})} + \|\phi_h\|_{L^p((0,T);W^{1,q})} \leq C_{p,q}(\|f\|_{L^p((0,T);L^q)} + \|\mathbf{g}\|_{L^p((0,T);L^q)}), \quad \text{when } \phi_h^0 \equiv 0, \quad (2.5)$$

$$\|\partial_t\phi_h\|_{L^p((0,T);L^q)} + \|\mathbf{A}_h\phi_h\|_{L^p((0,T);L^q)} \leq C_{p,q}\|f\|_{L^p((0,T);L^q)}, \quad \text{when } \phi_h^0 \equiv \mathbf{g} \equiv 0. \quad (2.6)$$

for  $1 < p, q < \infty$ .

The above  $L^p$  stability estimate and the maximal  $L^p$  regularity estimate were established in [18, 19] for  $A_{ij} = A_{ij}(x)$  being smooth and in [27] for  $A_{ij} = A_{ij}(x)$  being Lipschitz continuous. The inequalities (2.5)-(2.6) resemble the maximal  $L^p$  regularity of the continuous parabolic equation:

$$\|\partial_t\phi\|_{L^p((0,T);W^{-1,q})} + \|\phi\|_{L^p((0,T);W^{1,q})} \leq C_{p,q}(\|f\|_{L^p((0,T);L^q)} + \|\mathbf{g}\|_{L^p((0,T);L^q)}), \quad \text{when } \phi^0 \equiv 0, \quad (2.7)$$

$$\|\partial_t\phi\|_{L^p((0,T);L^q)} + \|\phi\|_{L^p((0,T);W^{2,q})} + \|\mathbf{A}\phi\|_{L^p((0,T);L^q)} \leq C_{p,q}\|f\|_{L^p((0,T);L^q)}, \quad \text{when } \phi^0 \equiv \mathbf{g} \equiv 0, \quad (2.8)$$

which were established in [23] for parabolic equations with Lipschitz continuous coefficients  $A_{ij} = A_{ij}(x)$ . (2.7)-(2.8) also hold for time-dependent Lipschitz continuous coefficients  $A_{ij} = A_{ij}(x, t)$  as a consequence of a simple perturbation argument.

**Corollary 2.1** Under the assumptions of Theorem 2.1, by choosing  $\phi_h^0$  as the Lagrangian interpolation of  $\phi$ , we have

$$\|\mathbf{P}_h\phi - \phi_h\|_{L^p((0,T);L^q)} \leq C_{p,q}(\|\phi^0\|_{W^{r+1,q}} + \|\phi\|_{L^p((0,T);W^{r+1,q})})h^{r+1}, \quad 1 < p, q < \infty, \quad (2.9)$$

$$\|\mathbf{P}_h\phi - \phi_h\|_{L^\infty((0,T);L^\infty)} \leq C(\|\phi^0\|_{W^{r+1,\infty}} + \|\phi\|_{L^\infty((0,T);W^{r+1,\infty})})h^{r+1-\epsilon_h}, \quad (2.10)$$

where  $\epsilon_h \in (0, 1)$  and satisfies  $\lim_{h \rightarrow 0} \epsilon_h = 0$ .

The proofs of Theorem 2.1 and Corollary 2.1 are presented in Section 4.

**Remark 2.1** By a transformation  $\phi = e^t \tilde{\phi}$ , it is easy to see that Theorem 2.1 and Corollary 2.1 also hold for parabolic equations without low-order terms, i.e.,

$$\begin{cases} \partial_t \phi - \nabla \cdot (A \nabla \phi) = f - \nabla \cdot \mathbf{g} & \text{in } \Omega, \\ A \nabla \phi \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n} & \text{on } \partial\Omega, \\ \phi(x, 0) = \phi_0(x) & \text{for } x \in \Omega. \end{cases}$$

Secondly, with the help of the above results, we study Galerkin FEMs for incompressible miscible flow in porous media with the Bear-Scheidegger dispersion model, which seeks  $P_h \in S_h^{r+1}/\{\text{constant}\}$  and  $c_h \in S_h^r$ ,  $n = 0, 1, \dots, N$ , such that the following equations hold for all  $\varphi_h \in S_h^{r+1}$  and  $\phi_h \in S_h^r$ :

$$\left( \frac{k(x)}{\mu(c_h)} \nabla P_h, \nabla \varphi_h \right) = (q_i - q_p, \varphi_h), \quad (2.11)$$

$$(\Phi \partial_t c_h, \phi_h) + (D(\mathbf{u}_h) \nabla c_h, \nabla \phi_h) + (\mathbf{u}_h \cdot \nabla c_h, \phi_h) = (\hat{c} q_i - c_h q_p, \phi_h), \quad (2.12)$$

with the initial condition  $c_h(0) = \Pi_h c(\cdot, 0)$ , where  $\Pi_h : C(\bar{\Omega}) \rightarrow S_h^r$  is the Lagrangian interpolation operator, and  $\mathbf{u}_h$  is given by

$$\mathbf{u}_h = -\frac{k(x)}{\mu(c_h)} \nabla P_h.$$

We present error estimates of Galerkin FEMs under the assumptions that  $q_i, q_p, \hat{c} \in L^\infty(\Omega_T)$ ,  $k \in W^{1,\infty}(\Omega)$ ,  $\mu \in W^{1,\infty}(\mathbb{R})$ ,  $k_0 \leq k(x) \leq k_1$  and  $\mu_0 \leq \mu(c) \leq \mu_1$  for some positive constants  $k_0$ ,  $k_1$ ,  $\mu_0$  and  $\mu_1$ .

**Theorem 2.2 (Optimal LP error estimate)** *For any given  $2 < p, q < \infty$  satisfying  $2/p + d/q < 1$ , if the solution of (1.1)-(1.4) exists and possesses the regularity*

$$P \in L^p((0, T); W^{r+2, q}), \quad \mathbf{u} \in L^\infty((0, T); W^{1, \infty}) \cap C(\bar{\Omega}_T), \quad (2.13)$$

$$c \in L^p((0, T); W^{r+1, q}), \quad c_0 \in W^{r+1, q}, \quad (2.14)$$

*then the finite element system (2.11)-(2.12) admits a unique solution  $(P_h, c_h)$  satisfying*

$$\|P_h - P\|_{L^p((0, T); W^{1, q})} + \|\mathbf{u}_h - \mathbf{u}\|_{L^p((0, T); L^q)} + \|c_h - c\|_{L^p((0, T); L^q)} \leq C_{p, q} h^{r+1}. \quad (2.15)$$

**Theorem 2.3 (Almost optimal  $L^\infty$  error estimate)** *If the solution of (1.1)-(1.4) exists and possesses the regularity*

$$\begin{aligned} P &\in L^p((0, T); W^{r+2, p}), & \mathbf{u} &\in L^\infty((0, T); W^{1, \infty}) \cap C(\bar{\Omega}_T), \\ c &\in L^p((0, T); W^{r+1, p}), & c_0 &\in W^{r+1, p}, \end{aligned}$$

*for all  $1 < p < \infty$ , then the finite element system (2.11)-(2.12) admits a unique solution  $(P_h, c_h)$  satisfying*

$$\|P_h - P\|_{L^\infty(\Omega_T)} + \|\mathbf{u}_h - \mathbf{u}\|_{L^\infty(\Omega_T)} + \|c_h - c\|_{L^\infty(\Omega_T)} \leq C h^{r+1-\epsilon_h}, \quad (2.16)$$

*where  $\epsilon_h \in (0, 1)$  and satisfies  $\lim_{h \rightarrow 0} \epsilon_h = 0$ .*

### 3 Proof of Theorems 2.2–2.3

In this section, we prove Theorem 2.2 and Theorem 2.3 based on Theorem 2.1. The proof of Theorem 2.1 is deferred to the next section.

#### 3.1 Preliminaries

The following lemma is concerned with the existence and continuity of the finite element solution. The proof will be given in Appendix.

**Lemma 3.1** *Under the assumption of Theorem 2.2, there exists a unique finite element solution  $P_h \in S_h^{r+1} \times [0, T]$  and  $c_h \in S_h^r \times [0, T]$  such that  $P_h \in L^\infty((0, T); W^{1,\infty})$  and  $c_h \in W^{1,\infty}(\Omega_T)$ , and*

$$\|\nabla c_h(t_1) - \nabla c_h(t_2)\|_{L^\infty} \leq C_h |t_1 - t_2| \quad \text{for } t_1, t_2 \in [0, T], \quad (3.1)$$

where  $C_h$  is independent of  $t_1$  and  $t_2$ .

We also need the following lemma as a generalization of Gronwall's inequality.

**Lemma 3.2** *Let  $1 < p < \infty$ . If the function  $Y \in C[0, T]$  satisfies*

$$\|Y\|_{L^p(\tau_1, \tau_2)} \leq \alpha \|Y\|_{L^1(\tau_1, \tau_2)} + \alpha Y(\tau_1) + \beta$$

for any  $0 \leq \tau_1 < \tau_2 \leq s$  and  $s \in (0, T]$ , with some positive constants  $\alpha$  and  $\beta$ , then we have

$$\|Y\|_{L^p(0, s)} \leq C_{T, \alpha, p}(Y(0) + \beta),$$

where the constant  $C_{T, \alpha, p}$  is independent of  $s \in (0, T]$ .

*Proof* By using Hölder's inequality, we obtain

$$\begin{aligned} \|Y\|_{L^p(\tau_1, \tau_2)} &\leq \alpha \|Y\|_{L^1(\tau_1, \tau_2)} + \alpha Y(\tau_1) + \beta \\ &\leq \alpha(\tau_2 - \tau_1)^{1-1/p} \|Y\|_{L^p(\tau_1, \tau_2)} + \alpha Y(\tau_1) + \beta. \end{aligned}$$

Choose  $\Delta T < 1/(2\alpha)^{1/(1-1/p)}$  and divide the interval  $[0, s]$  into  $0 = T_0 < T_1 < T_2 < \dots < T_N = s$  in the following way. If  $T_k + \Delta T < s$ , then we apply the above inequality to get

$$\|Y\|_{L^p(T_k, T_k + \Delta T)} \leq C_{T, \alpha, p} Y(T_k) + C_{T, \alpha, p} \beta.$$

Then we choose  $T_{k+1} \in [T_k + \Delta T/2, T_k + \Delta T]$  satisfying  $Y(T_{k+1}) \leq \frac{2^{1/p}}{\Delta T^{1/p}} \|Y\|_{L^p(T_k, T_k + \Delta T)}$  (as a consequence of the mean value theorem) so that

$$Y(T_{k+1}) \leq C_{T, \alpha, p} Y(T_k) + C_{T, \alpha, p} \beta \quad \text{and} \quad \Delta T/2 \leq T_{k+1} - T_k \leq \Delta T.$$

Iterations of the above two inequalities give  $\|Y\|_{L^p(0, s)} \leq C_{T, \alpha, p}(Y(0) + \beta)$ . ■

The following lemma concerns the boundedness of solution for parabolic equations [2, 3].

**Lemma 3.3 (De Giorgi–Nash–Moser)** *If  $1 < p, q < \infty$  and  $2/p + d/q < 1$ , then the solution of the parabolic equation (1.11) satisfies that*

$$\|\phi\|_{L^\infty(\Omega_T)} \leq C_{p, q, T} (\|\phi_0\|_{L^\infty} + \|f\|_{L^p((0, T); L^q)} + \|\mathbf{g}\|_{L^p((0, T); L^q)}).$$

For fixed  $p$  and  $q$ , the constant  $C_{p, q, T}$  is bounded if  $T$  is bounded.

### 3.2 Proof of Theorem 2.2

Before we start proving Theorem 2.2, we study the following inequality

$$\|\nabla c_h(t)\|_{L^\infty} \leq \|\nabla \mathbf{P}_h c(t)\|_{L^\infty} + 1, \quad (3.2)$$

which clearly holds for  $t = 0$  when  $h < h_0$ , for some positive constant  $h_0$ . If we assume that the inequality holds for  $t \in [0, s]$ , where  $s$  is some nonnegative number, then by the continuity of solution given in Lemma 3.1 we can assume that

$$\|\nabla c_h(t)\|_{L^\infty} \leq \|\nabla \mathbf{P}_h c(t)\|_{L^\infty} + 2, \quad (3.3)$$

for  $t \in [0, s + \delta_h]$ , where  $\delta_h$  is some positive constant. We proceed to prove that (3.2) holds for  $t \in [0, s + \delta_h]$  so that by iterations we can derive that (3.2) holds for all  $t \in [0, T]$ .

We let  $\tau_1$  and  $\tau_2$  be two given positive constants, satisfying  $0 \leq \tau_1 < \tau_2 \leq s + \delta_h$ , and keep in mind that all the generic constants below are independent of  $\tau_1$  and  $\tau_2$ .

Also we let  $\theta \in L^2((\tau_1, \tau_2); H^1) \cap H^1((\tau_1, \tau_2); H^{-1})$  be the solution of the auxiliary parabolic equation:

$$\begin{aligned} & \partial_t \theta - \nabla \cdot (D(\mathbf{u}) \nabla \theta) + \theta \\ &= \nabla \cdot [(D(\mathbf{u}_h) - D(\mathbf{u})) \nabla c_h] - \nabla \cdot [\mathbf{u}(c_h - c)] - (\mathbf{u}_h - \mathbf{u}) \cdot \nabla c_h + (c_h - c)(1 - q_p) \end{aligned} \quad (3.4)$$

with the boundary condition  $-D(\mathbf{u}) \nabla \theta \cdot \mathbf{n} = (D(\mathbf{u}_h) - D(\mathbf{u})) \nabla c_h \cdot \mathbf{n}$  and the initial condition  $\theta(\tau_1) = 0$ . Its finite element solution  $\{\theta_h(t)\}_{t \in (\tau_1, \tau_2)}$  is defined by

$$\begin{aligned} & (\partial_t \theta_h, \phi_h) + (D(\mathbf{u}) \nabla \theta_h, \nabla \phi_h) + (\theta_h, \phi_h) \\ &= (- (D(\mathbf{u}_h) - D(\mathbf{u})) \nabla c_h + \mathbf{u}(c_h - c), \nabla \phi_h) \\ & \quad - ((\mathbf{u}_h - \mathbf{u}) \cdot \nabla c_h + (c_h - c)(1 - q_p), \phi_h), \quad \forall \phi_h \in S_h^r, \end{aligned} \quad (3.5)$$

with the initial condition  $\theta_h(\tau_1) = 0$ . Since the exact solution  $(P, c)$  satisfies that

$$\left( \frac{k(x)}{\mu(c)} \nabla P, \nabla \varphi \right) = (q_i - q_p, \varphi), \quad \forall \varphi \in H^1, \quad (3.6)$$

$$(\Phi \partial_t c, \phi) + (D(\mathbf{u}) \nabla c, \nabla \phi) + (c, \phi) = (\hat{c} q_i + c(1 - q_p) - \mathbf{u} \cdot \nabla c, \phi), \quad \forall \phi \in H^1, \quad (3.7)$$

by comparing (2.11)-(2.12) with (3.6)-(3.7) we derive that

$$\begin{aligned} \left( \frac{k(x)}{\mu(c_h)} \nabla (P_h - \mathbf{P}_h P), \nabla \varphi_h \right) &= \left( \frac{k(x)}{\mu(c_h)} \nabla (P - \mathbf{P}_h P), \nabla \varphi_h \right) \\ & \quad + \left( \left( \frac{k(x)}{\mu(c)} - \frac{k(x)}{\mu(c_h)} \right) \nabla P, \nabla \varphi_h \right), \quad \forall \varphi_h \in S_h^{r+1}, \end{aligned} \quad (3.8)$$

$$\begin{aligned} & (\Phi \partial_t (c_h - \theta_h), \phi_h) + (D(\mathbf{u}) \nabla (c_h - \theta_h), \nabla \phi_h) + (c_h - \theta_h, \phi_h) \\ &= (\hat{c} q_i + c(1 - q_p) - \mathbf{u} \cdot \nabla c, \phi_h), \quad \forall \phi_h \in S_h^r. \end{aligned} \quad (3.9)$$

Here the solution  $c_h - \theta_h$  of the last equation can be viewed as the FE approximation to the equation (3.7). Therefore, we apply Theorem 2.1 to these two equations to obtain

$$\|c_h - \theta_h - \mathbf{P}_h c\|_{L^p((\tau_1, \tau_2); L^q)} \leq C_{p,q} \|\mathbf{P}_h c(\tau_1) - c_h(\tau_1)\|_{L^q} + C_{p,q} \|c\|_{L^p((\tau_1, \tau_2); W^{r+1,q})} h^{r+1}. \quad (3.10)$$



To estimate  $\mathbf{P}_h c - c_h$ , we need to estimate  $\theta_h$ . Again, applying Theorem 2.1 to the equations (3.4)-(3.5) we get

$$\begin{aligned}\|\theta_h\|_{L^p((\tau_1, \tau_2); L^q)} &\leq \|\theta - \theta_h\|_{L^p((\tau_1, \tau_2); L^q)} + \|\theta\|_{L^p((\tau_1, \tau_2); L^q)} \\ &\leq C_{p,q} \|\mathbf{P}_h \theta - \mathbf{R}_h \theta\|_{L^p((\tau_1, \tau_2); L^q)} + C_{p,q} \|\theta - \mathbf{P}_h \theta\|_{L^p((\tau_1, \tau_2); L^q)} + \|\theta\|_{L^p((\tau_1, \tau_2); L^q)} \\ &\leq C_{p,q} h \|\theta\|_{L^p((\tau_1, \tau_2); W^{1,q})} + \|\theta\|_{L^p((\tau_1, \tau_2); L^q)}\end{aligned}$$

for any  $\tau_1, \tau_2 \in [0, s + \delta_h]$  and  $2/p + d/q < 1$ . According to Lemma 3.3, (3.4) implies that, by choosing  $p_0 \in (2, p)$  satisfying  $2/p_0 + d/q < 1$  and noting the fact  $\theta(\tau_1) = 0$ ,

$$\begin{aligned}\|\theta\|_{L^\infty((\tau_1, \tau_2); L^\infty)} &\leq C_{p_0,q} \| (D(\mathbf{u}_h) - D(\mathbf{u})) \nabla c_h \|_{L^{p_0}((\tau_1, \tau_2); L^q)} \\ &\quad + C_{p,q} \| (\mathbf{u}_h - \mathbf{u}) \cdot \nabla c_h \|_{L^{p_0}((\tau_1, \tau_2); L^q)} + \|c_h - c\|_{L^{p_0}((\tau_1, \tau_2); L^q)} \\ &\leq C_{p_0,q} \|\nabla c_h\|_{L^\infty(\Omega_{\tau_2})} \|\mathbf{u}_h - \mathbf{u}\|_{L^{p_0}((\tau_1, \tau_2); L^q)} + C_{p_0,q} \|c_h - c\|_{L^{p_0}((\tau_1, \tau_2); L^q)}\end{aligned}$$

and by using (2.5) and (2.7),

$$\begin{aligned}\|\partial_t \theta\|_{L^p((\tau_1, \tau_2); W^{-1,q})} &+ \|\theta\|_{L^p((\tau_1, \tau_2); W^{1,q})} + \|\theta_h\|_{L^p((\tau_1, \tau_2); W^{1,q})} \\ &\leq C_{p,q} \| (D(\mathbf{u}_h) - D(\mathbf{u})) \nabla c_h \|_{L^p((\tau_1, \tau_2); L^q)} \\ &\quad + C_{p,q} \| (\mathbf{u}_h - \mathbf{u}) \cdot \nabla c_h \|_{L^p((\tau_1, \tau_2); L^q)} + \|c_h - c\|_{L^p((\tau_1, \tau_2); L^q)} \\ &\leq C_{p,q} \|\nabla c_h\|_{L^\infty(\Omega_{\tau_2})} \|\mathbf{u}_h - \mathbf{u}\|_{L^p((\tau_1, \tau_2); L^q)} + C_{p,q} \|c_h - c\|_{L^p((\tau_1, \tau_2); L^q)}.\end{aligned}\tag{3.11}$$

The last three inequalities imply that

$$\begin{aligned}\|\theta_h\|_{L^p((\tau_1, \tau_2); L^q)} &\leq C_{p_0,q} (\|\nabla c_h\|_{L^\infty(\Omega_{\tau_2})} \|\mathbf{u}_h - \mathbf{u}\|_{L^{p_0}((\tau_1, \tau_2); L^q)} + \|c_h - c\|_{L^{p_0}((\tau_1, \tau_2); L^q)}) \\ &\quad + C_{p,q} h (\|\nabla c_h\|_{L^\infty(\Omega_{\tau_2})} \|\mathbf{u}_h - \mathbf{u}\|_{L^p((\tau_1, \tau_2); L^q)} + \|c_h - c\|_{L^p((\tau_1, \tau_2); L^q)}).\end{aligned}\tag{3.12}$$

Combining the inequalities (3.10) and (3.12) gives

$$\begin{aligned}\|c_h - \mathbf{P}_h c\|_{L^p((\tau_1, \tau_2); L^q)} &\leq C_{p,q} \|\mathbf{P}_h c(\tau_1) - c_h(\tau_1)\|_{L^q} \\ &\quad + C_{p,q} (h^{r+1} + \|c_h - c\|_{L^{p_0}((\tau_1, \tau_2); L^q)} + h \|c_h - c\|_{L^p((\tau_1, \tau_2); L^q)}) \\ &\quad + C_{p,q} \|\nabla c_h\|_{L^\infty(\Omega_{\tau_2})} (\|\mathbf{u}_h - \mathbf{u}\|_{L^{p_0}((\tau_1, \tau_2); L^q)} + h \|\mathbf{u}_h - \mathbf{u}\|_{L^p((\tau_1, \tau_2); L^q)}),\end{aligned}\tag{3.13}$$

which together with (3.3) implies that, when  $h < 1/(2C_{p,q})$ ,

$$\begin{aligned}\|c_h - \mathbf{P}_h c\|_{L^p((\tau_1, \tau_2); L^q)} &\leq C_{p,q} \|\mathbf{P}_h c(\tau_1) - c_h(\tau_1)\|_{L^q} + C_{p,q} (h^{r+1} + \|c_h - c\|_{L^{p_0}((\tau_1, \tau_2); L^q)}) \\ &\quad + C_{p,q} (\|\mathbf{u}_h - \mathbf{u}\|_{L^{p_0}((\tau_1, \tau_2); L^q)} + h \|\mathbf{u}_h - \mathbf{u}\|_{L^p((\tau_1, \tau_2); L^q)}).\end{aligned}\tag{3.14}$$

By applying the  $W^{1,q}$  estimate (2.3) to the equation (3.8), we see that

$$\begin{aligned}\|P_h - \mathbf{P}_h P\|_{W^{1,q}} + \|P_h - P\|_{W^{1,q}} &\leq C_q \|c - c_h\|_{L^q} + C_q \|P - \mathbf{P}_h P\|_{W^{1,q}}, \\ &\leq C_q \|c - c_h\|_{L^q} + C_q \|P\|_{W^{r+2,q}} h^{r+1},\end{aligned}\tag{3.15}$$

and by an inverse inequality, we derive that

$$\begin{aligned}\|P_h - \mathbf{P}_h P\|_{W^{1,\infty}} &\leq C h^{-d/q} \|P_h - \mathbf{P}_h P\|_{W^{1,q}} \\ &\leq C_q h^{-d/q} (\|c - c_h\|_{L^q} + \|P\|_{W^{r+2,q}} h^{r+1}).\end{aligned}\tag{3.16}$$

Therefore, we have

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^q} \leq C_q \|P\|_{W^{r+2,q}} h^{r+1} + C_q \|c - c_h\|_{L^q}, \quad (3.17)$$

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^\infty} \leq C_q h^{-d/q} (\|P\|_{W^{r+2,q}} h^{r+1} + \|c - c_h\|_{L^q}) + \|c - c_h\|_{L^\infty}. \quad (3.18)$$

Furthermore, from (3.14) and (3.17), we derive that

$$\begin{aligned} & \|\mathbf{P}_h c - c_h\|_{L^p((\tau_1, \tau_2); L^q)} \\ & \leq C_{p,q} \|\mathbf{P}_h c(\tau_1) - c_h(\tau_1)\|_{L^q} + C_{p,q} h^{r+1} + C_{p,q} h \|c_h - c\|_{L^p((\tau_1, \tau_2); L^q)} + C_{p,q} \|c_h - c\|_{L^{p_0}((\tau_1, \tau_2); L^q)} \\ & \leq C_{p,q} \|\mathbf{P}_h c(\tau_1) - c_h(\tau_1)\|_{L^q} + C_{p,q} h^{r+1} + C_{p,q} h \|c_h - c\|_{L^p((\tau_1, \tau_2); L^q)} \\ & \quad + \epsilon \|c_h - c\|_{L^p((\tau_1, \tau_2); L^q)} + C_{p,q,\epsilon} \|c_h - c\|_{L^1((\tau_1, \tau_2); L^q)}, \end{aligned}$$

which further reduces to

$$\|\mathbf{P}_h c - c_h\|_{L^p((\tau_1, \tau_2); L^q)} \leq C_{p,q} \|c_h - c\|_{L^1((\tau_1, \tau_2); L^q)} + C_{p,q} \|\mathbf{P}_h c(\tau_1) - c_h(\tau_1)\|_{L^q} + C_{p,q} h^{r+1}$$

when  $h < 1/(2C_{p,q})$ . Applying Lemma 3.2 to the above inequality with (3.15) and (3.17) leads to

$$\|c_h - c\|_{L^p((0, \tau); L^q)} + \|P_h - P\|_{L^p((0, \tau); W^{1,q})} + \|\mathbf{u}_h - \mathbf{u}\|_{L^p((0, \tau); L^q)} \leq C_{p,q} h^{r+1} \quad (3.19)$$

for  $\tau = s + \delta_h$ , where the constant  $C_{p,q}$  is independent of  $s$  and  $\delta_h$  (but may depend on  $T$ ).

From (3.11) and the above inequality we also see that

$$\|\partial_t \theta\|_{L^p((0, \tau); W^{-1,q})} + \|\theta\|_{L^p((0, \tau); W^{1,q})} + \|\theta_h\|_{L^p((0, \tau); W^{1,q})} \leq C_{p,q} h^{r+1} \quad (3.20)$$

for  $\tau = s + \delta_h$ , and from the equation (3.5),

$$\begin{aligned} \int_0^\tau (\partial_t \theta_h, \phi) dt &= \int_0^\tau (\partial_t \theta_h, \mathbf{P}_h \phi) dt \\ &= \int_0^\tau (\partial_t \theta, \mathbf{P}_h \phi) dt - \int_0^\tau (\partial_t (\theta - \theta_h), \mathbf{P}_h \phi) dt \\ &= \int_0^\tau (\partial_t \theta, \mathbf{P}_h \phi) dt + \int_0^\tau (D(\mathbf{u}) \nabla (\theta - \theta_h), \nabla \mathbf{P}_h \phi) dt \\ &\leq C (\|\partial_t \theta\|_{L^p((0, \tau); W^{-1,q})} + \|\theta - \theta_h\|_{L^p((0, \tau); W^{1,q})}) \|\mathbf{P}_h \phi\|_{L^{p'}((0, \tau); W^{1,q'})} \\ &\leq C (\|\partial_t \theta\|_{L^p((0, \tau); W^{-1,q})} + \|\theta - \theta_h\|_{L^p((0, \tau); W^{1,q})}) \|\phi\|_{L^{p'}((0, \tau); W^{1,q'})} \end{aligned}$$

for any  $\phi \in L^{p'}((0, \tau); W^{1,q'})$ , which in turn produces

$$\|\partial_t \theta_h\|_{L^p((0, \tau); W^{-1,q})} \leq C_{p,q} h^{r+1}. \quad (3.21)$$

Note that the difference between (3.7) and (3.9) gives

$$\left( \Phi \partial_t (c_h - \theta_h - \mathbf{P}_h c), \phi_h \right) + \left( D(\mathbf{u}) \nabla (c_h - \theta_h - c), \nabla \phi_h \right) + (c_h - \theta_h - c, \phi_h) = 0, \quad \forall \phi_h \in S_h^r,$$

which leads to

$$\|\partial_t (c_h - \mathbf{P}_h c)\|_{L^p((0, \tau); W^{-1,q})}$$

$$\begin{aligned}
&\leq C(\|c_h - c\|_{L^p((0,\tau);W^{1,q})} + \|\theta_h\|_{L^p((0,\tau);W^{1,q})} + \|\partial_t \theta_h\|_{L^p((0,\tau);W^{-1,q})}) \\
&\leq C(h^{-1}\|c_h - c\|_{L^p((0,\tau);L^q)} + \|\theta_h\|_{L^p((0,\tau);W^{1,q})} + \|\partial_t \theta_h\|_{L^p((0,\tau);W^{-1,q})}) \\
&\leq C_{p,q} h^r.
\end{aligned}$$

By inverse inequalities and interpolation inequalities, we obtain

$$\begin{aligned}
\|\partial_t(c_h - \mathbf{P}_h c)\|_{L^p((0,\tau);L^\infty)} &\leq C h^{-1-d/q} \|\partial_t(c_h - \mathbf{P}_h c)\|_{L^p((0,\tau);W^{-1,q})} \leq C_{p,q} h^{r-1-d/q}, \\
\|c_h - \mathbf{P}_h c\|_{L^p((0,\tau);L^\infty)} &\leq C h^{-d/q} \|c_h - \mathbf{P}_h c\|_{L^p((0,\tau);L^q)} \leq C_{p,q} h^{r+1-d/q},
\end{aligned}$$

$$\begin{aligned}
&\|c_h - \mathbf{P}_h c\|_{L^\infty((0,\tau);L^\infty)} \\
&\leq C \|c_h(0) - \mathbf{P}_h c(0)\|_{L^\infty} + C \|c_h - \mathbf{P}_h c\|_{L^p((0,\tau);L^\infty)}^{1-1/p} \|\partial_t(c_h - \mathbf{P}_h c)\|_{L^p((0,\tau);L^\infty)}^{1/p} \\
&\leq C \|c(0)\|_{W^{r+1,q}} h^{r+1-d/q} + C_{p,q} h^{r+1-2/p-d/q},
\end{aligned} \tag{3.22}$$

and

$$\|\nabla(c_h - \mathbf{P}_h c)\|_{L^\infty((0,\tau);L^\infty)} \leq C_{p,q} h^{r-d/q} + C_{p,q} h^{r-2/p-d/q}. \tag{3.23}$$

When  $2/p + d/q < 1$ , (3.23) implies the existence of a positive constant  $h_1 < h_0$  (independent of  $s$  and  $\delta_h$ ) such that (3.2) holds for  $t \in [0, s + \delta_h]$  when  $h < h_1$ . By induction, (3.2) holds for all  $t \in [0, T]$  and therefore, (3.15)-(3.23) hold for  $\tau = T$  with the same constants  $C_{p,q}$ . Thus the theorem is proved when  $h < h_1$ .

When  $h \geq h_1$ , we substitute  $\varphi_h = P_h$  in (2.11) to get

$$\|P_h\|_{L^\infty((0,T);H^1)} + \|\mathbf{u}_h\|_{L^\infty((0,T);L^2)} \leq C,$$

which implies that, by an inverse inequality (since  $q > d \geq 2$ ),

$$\|P_h\|_{L^\infty((0,T);W^{1,q})} + \|\mathbf{u}_h\|_{L^\infty((0,T);L^q)} \leq C h_1^{d/q-d/2}.$$

Then we substitute  $\phi_h = c_h$  in (2.12) and get  $\|c_h\|_{L^\infty((0,T);L^2)} \leq C_{h_1}$ , which with an inverse inequality again leads to

$$\|c_h\|_{L^\infty((0,T);L^q)} \leq C_{h_1} h_1^{d/q-d/2}.$$

By the inverse inequality, the last two inequalities show that

$$\|P_h - P\|_{L^\infty((0,T);W^{1,\infty})} + \|\mathbf{u}_h - \mathbf{u}\|_{L^\infty((0,T);L^\infty)} + \|c_h - c\|_{L^\infty((0,T);L^\infty)} \leq C_{h_1} \leq C_{h_1} h_1^{-r-1} h^{r+1}.$$

This proves (2.15) for  $h \geq h_1$ .

The proof of Theorem 2.2 is completed. ■

Theorem 2.3 is a simple consequence of Theorem 2.2 (see the proof of Corollary 2.1 in the next section).

## 4 Proof of Theorem 2.1 and Corollary 2.1

In this section, we prove Theorem 2.1 and Corollary 2.1, which have been used in the last section to prove Theorem 2.2 and Theorem 2.3.

#### 4.1 Proof of Theorem 2.1

First, we consider the case  $\phi_0 = \phi_h^0 = 0$  and rewrite (1.12) by

$$\begin{cases} \partial_t \phi_h(t) + \mathbf{A}_h(t) \phi_h(t) = \mathbf{P}_h f(\cdot, t) - \bar{\nabla}_h \cdot \mathbf{g}(\cdot, t), \\ \phi_h(0) = 0. \end{cases} \quad (4.1)$$

Let  $\psi_h(t) = \mathbf{P}_h \phi(t) - \phi_h(t)$  so that  $\psi_h$  is the solution of the following equation:

$$\begin{cases} \partial_t \psi_h(t) + \mathbf{A}_h(t_n) \psi_h(t) = (\mathbf{A}_h(t_n) - \mathbf{A}_h(t)) \psi_h(t) + \mathbf{A}_h(t) (\mathbf{P}_h \phi(t) - \phi(t)), \\ \psi_h(0) = 0. \end{cases}$$

We divide the interval  $[0, T]$  into  $0 = t_0 < t_1 < \dots < t_N = T$  uniformly with  $t_n - t_{n-1} = \Delta t$  for  $1 \leq n \leq N$ , and let  $\varphi_h^n(t) = \psi_h(t) - \psi_h(2t_n - t)$  for  $t \in [t_n, t_{n+1}]$ . Then  $\varphi_h^n$  is the solution to the equation

$$\begin{aligned} & \partial_t \varphi_h^n(t) + \mathbf{A}_h(t_n) \varphi_h^n(t) \\ &= (\mathbf{A}_h(t_n) - \mathbf{A}_h(t)) \psi_h(t) + \mathbf{A}_h(t) (\mathbf{P}_h \phi(t) - \phi(t)) - \mathbf{A}_h(t_n) \psi_h(2t_n - t) + \partial_t \psi_h(2t_n - t), \\ &= (\mathbf{A}_h(t_n) - \mathbf{A}_h(t)) \psi_h(t) + \mathbf{A}_h(t) (\mathbf{P}_h \phi(t) - \mathbf{R}_h(t) \phi(t)) - \mathbf{A}_h(t_n) \psi_h(2t_n - t) \\ &\quad - \mathbf{A}_h(2t_n - t) \psi_h(2t_n - t) + \mathbf{A}_h(2t_n - t) (\mathbf{P}_h \phi(2t_n - t) - \mathbf{R}_h(2t_n - t) \phi(2t_n - t)) \end{aligned}$$

for  $t \in [t_n, t_{n+1}]$  with the initial condition  $\varphi_h^n(t_n) = 0$ . Since (2.5) holds when the coefficient matrix  $A$  is independent of  $t$  [27], we apply the inequality (2.5) to the above equation to get

$$\begin{aligned} \|\varphi_h^n\|_{L^p((t_n, t_{n+1}); W^{1,q})} &\leq C \sup_{t \in [t_n, t_{n+1}]} \|A_{ij}(\cdot, t_n) - A_{ij}(\cdot, t)\|_{L^\infty} \|\psi_h\|_{L^p((t_n, t_{n+1}); W^{1,q})} \\ &\quad + C \|\psi_h\|_{L^p((t_{n-1}, t_n); W^{1,q})} + C \|\mathbf{P}_h \phi - \mathbf{R}_h \phi\|_{L^p((t_{n-1}, t_{n+1}); W^{1,q})}. \end{aligned}$$

Since  $A_{ij} \in C(\bar{\Omega}_T)$ , by choosing  $\Delta t$  small enough we have

$$C \sup_{t \in [t_n, t_{n+1}]} \|A_{ij}(\cdot, t_n) - A_{ij}(\cdot, t)\|_{L^\infty} < 1/2$$

and so

$$\|\psi_h\|_{L^p((t_n, t_{n+1}); W^{1,q})} \leq C \|\psi_h\|_{L^p((t_{n-1}, t_n); W^{1,q})} + C \|\mathbf{P}_h \phi - \mathbf{R}_h \phi\|_{L^p((t_{n-1}, t_{n+1}); W^{1,q})}.$$

Iterating the above inequality gives

$$\|\psi_h\|_{L^p((0, T); W^{1,q})} \leq C \|\mathbf{P}_h \phi - \mathbf{R}_h \phi\|_{L^p((0, T); W^{1,q})}, \quad (4.2)$$

which implies that

$$\begin{aligned} \|\phi_h\|_{L^p((0, T); W^{1,q})} &\leq C \|\psi_h\|_{L^p((0, T); W^{1,q})} + C \|\mathbf{P}_h \phi\|_{L^p((0, T); W^{1,q})} \\ &\leq C \|\mathbf{P}_h \phi - \mathbf{R}_h \phi\|_{L^p((0, T); W^{1,q})} + C \|\mathbf{P}_h \phi\|_{L^p((0, T); W^{1,q})}, \\ &\leq C \|\phi\|_{L^p((0, T); W^{1,q})} \\ &\leq C_{p,q} (\|f\|_{L^p((0, T); L^q)} + \|\mathbf{g}\|_{L^p((0, T); L^q)}), \end{aligned}$$

where we have used (2.2) and (2.7). The proof of (2.5) is completed.

Secondly, we prove (2.6) when  $\phi^0 \equiv \phi_h^0 \equiv \mathbf{g} \equiv 0$ . Note that

$$\begin{aligned} |(\mathbf{A}_h(t)\psi_h(t), v)| &= |(\mathbf{A}_h(t)\psi_h(t), \mathbf{P}_h v)| = |(A(t)\nabla\psi_h(t), \nabla\mathbf{P}_h v)| \\ &\leq C\|\psi_h(t)\|_{W^{1,q}}\|\mathbf{P}_h v\|_{W^{1,q'}} \leq Ch^{-1}\|\psi_h(t)\|_{W^{1,q}}\|\mathbf{P}_h v\|_{L^{q'}} \leq Ch^{-1}\|\psi_h(t)\|_{W^{1,q}}\|v\|_{L^{q'}}, \end{aligned}$$

which shows that

$$\|\mathbf{A}_h(t)\psi_h(t)\|_{L^q} \leq Ch^{-1}\|\psi_h(t)\|_{W^{1,q}} \quad (4.3)$$

and, as a consequence of (4.2)-(4.3),

$$\begin{aligned} \|\mathbf{A}_h\phi_h\|_{L^p((0,T);L^q)} &\leq \|\mathbf{A}_h\mathbf{P}_h\phi\|_{L^p((0,T);L^q)} + \|\mathbf{A}_h\psi_h\|_{L^p((0,T);L^q)} \\ &\leq \|\mathbf{A}_h\mathbf{P}_h\phi\|_{L^p((0,T);L^q)} + Ch^{-1}\|\psi_h\|_{L^p((0,T);W^{1,q})} \\ &\leq \|\mathbf{A}_h\mathbf{P}_h\phi\|_{L^p((0,T);L^q)} + Ch^{-1}\|\mathbf{R}_h\phi - \mathbf{P}_h\phi\|_{L^p((0,T);W^{1,q})} \\ &\leq \|\mathbf{A}_h\mathbf{P}_h\phi\|_{L^p((0,T);L^q)} + C\|\phi\|_{L^p((0,T);W^{2,q})}. \end{aligned}$$

It suffices to prove that  $\|\mathbf{A}_h\mathbf{P}_h\phi\|_{L^p((0,T);L^q)} \leq C\|f\|_{L^p((0,T);L^q)}$ , which is a consequence of

$$\begin{aligned} |(\mathbf{A}_h(t)\mathbf{P}_h\phi(t), v)| &= |(\mathbf{A}_h(t)\mathbf{P}_h\phi(t), \mathbf{P}_h v)| \\ &\leq |(\mathbf{A}_h(t)(\mathbf{P}_h\phi(t) - \mathbf{R}_h(t)\phi(t)), \mathbf{P}_h v)| + |(\mathbf{A}(t)\phi(t), \mathbf{P}_h v)| \\ &\leq C\|\mathbf{P}_h\phi(t) - \mathbf{R}_h(t)\phi(t)\|_{W^{1,q}}\|\mathbf{P}_h v\|_{W^{1,q'}} + \|\mathbf{A}(t)\phi(t)\|_{L^q}\|\mathbf{P}_h v\|_{L^{q'}} \\ &\leq C(\|\phi(t)\|_{W^{2,q}} + \|\mathbf{A}(t)\phi(t)\|_{L^q})\|v\|_{L^{q'}} \end{aligned}$$

and (2.8). Then from (4.1) we derive that

$$\|\partial_t\phi_h\|_{L^p((0,T);L^q)} \leq \|\mathbf{A}_h\phi_h\|_{L^p((0,T);L^q)} + \|\mathbf{P}_h f\|_{L^p((0,T);L^q)} \leq C_{p,q}\|f\|_{L^p((0,T);L^q)}.$$

Finally, we prove (2.4) by considering  $e_h(t) = \mathbf{P}_h\phi(t) - \phi_h(t) + \phi_h^0 - \mathbf{P}_h\phi_0$ , which is the solution of

$$\begin{cases} \partial_t e_h(t) + \mathbf{A}_h(t)e_h(t) = \mathbf{A}_h(t)g_h(t), \\ \phi_h(0) = 0. \end{cases} \quad (4.4)$$

where  $g_h(t) = \phi_h^0 - \mathbf{P}_h\phi_0 + \mathbf{P}_h\phi(t) - \mathbf{R}_h(t)\phi(t)$ . By using (2.5), we obtain

$$\|\partial_t e_h\|_{L^p((0,T);W^{-1,q})} + \|e_h\|_{L^p((0,T);W^{1,q})} \leq C_{p,q}\|g_h\|_{L^p((0,T);W^{1,q})},$$

which further implies that

$$\begin{aligned} \|\partial_t(\mathbf{P}_h\phi - \phi_h)\|_{L^p((0,T);W^{-1,q})} &+ \|\mathbf{P}_h\phi - \phi_h\|_{L^p((0,T);W^{1,q})} \\ &\leq C_{p,q}(\|g_h\|_{L^p((0,T);W^{1,q})} + \|\phi_h^0 - \mathbf{P}_h\phi_0\|_{W^{1,q}}). \end{aligned} \quad (4.5)$$

Let  $\bar{g}_h(t) = \mathbf{P}_h\phi(t) - \mathbf{R}_h(t)\phi(t)$  and let  $v$  be the solution of the backward parabolic equation

$$\begin{cases} \partial_t v + \nabla \cdot (A\nabla v) - v = -\varphi & \text{in } \Omega, \\ A\nabla v \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ v(T) = 0, \end{cases} \quad (4.6)$$

which leads to a basic estimate

$$\|v(0)\|_{L^{q'}} \leq C\|\varphi\|_{L^1((0,T);L^{q'})} \leq C\|\varphi\|_{L^{p'}((0,T);L^{q'})}. \quad (4.7)$$

We see that  $w_h(t) = \mathbf{P}_h\phi(t) - \phi_h(t)$  satisfies

$$\begin{aligned} & \int_0^T (w_h, \varphi) \, dt \\ &= \int_0^T (w_h, -\partial_t v - \nabla \cdot (A \nabla v) + v) \, dt \\ &= \int_0^T [(\partial_t w_h, v) + (A \nabla w_h, \nabla v) + (w_h, v)] \, dt + (\mathbf{P}_h \phi^0 - \phi_h^0, v(0)) \\ &= \int_0^T [(\partial_t w_h(t), v(t) - \mathbf{P}_h v(t)) + (\mathbf{A}_h(t) w_h, v(t) - \mathbf{P}_h v(t))] \, dt \\ &\quad + (\mathbf{P}_h \phi^0 - \phi_h^0, v(0)) + \int_0^T (\mathbf{A}_h(t) \bar{g}_h(t), \mathbf{P}_h v(t)) \, dt \\ &= \int_0^T (\mathbf{A}_h(t) w_h(t), \mathbf{R}_h(t) v(t) - \mathbf{P}_h v(t)) \, dt + (\mathbf{P}_h \phi^0 - \phi_h^0, v(0)) \\ &\quad + \int_0^T (\bar{g}_h(t), \mathbf{A}_h(t) (\mathbf{P}_h v(t) - \mathbf{R}_h(t) v(t))) \, dt + \int_0^T (\bar{g}_h(t), \mathbf{A}(t) v(t)) \, dt \\ &\leq C \|w_h\|_{L^p((0,T);W^{1,q})} \|\mathbf{R}_h v - \mathbf{P}_h v\|_{L^{p'}((0,T);W^{1,q'})} + \|\mathbf{P}_h \phi^0 - \phi_h^0\|_{L^q} \|v(0)\|_{L^{q'}} \\ &\quad + C \|\bar{g}_h\|_{L^p((0,T);W^{1,q})} \|\mathbf{R}_h v - \mathbf{P}_h v\|_{L^{p'}((0,T);W^{1,q'})} + \|\bar{g}_h\|_{L^p((0,T);L^q)} \|\mathbf{A} v\|_{L^{p'}((0,T);L^{q'})} \\ &\leq C_{p,q} h \|w_h\|_{L^p((0,T);W^{1,q})} \|v\|_{L^{p'}((0,T);W^{2,q'})} + \|\mathbf{P}_h \phi^0 - \phi_h^0\|_{L^q} \|v(0)\|_{L^{q'}} \\ &\quad + C \|\bar{g}_h\|_{L^p((0,T);L^q)} (\|v\|_{L^{p'}((0,T);W^{2,q'})} + \|\mathbf{A} v\|_{L^{p'}((0,T);L^{q'})}) \\ &\leq C_{p,q} (h \|w_h\|_{L^p((0,T);W^{1,q})} + \|\bar{g}_h\|_{L^p((0,T);L^q)} + \|\mathbf{P}_h \phi^0 - \phi_h^0\|_{L^q}) \|\varphi\|_{L^{p'}((0,T);L^{q'})}, \end{aligned}$$

where we have used (4.7). By duality and using (4.5), we derive that

$$\begin{aligned} \|w_h\|_{L^p((0,T);L^q)} &\leq C_{p,q} (h \|w_h\|_{L^p((0,T);W^{1,q})} + \|\bar{g}_h\|_{L^p((0,T);L^q)} + \|\mathbf{P}_h \phi^0 - \phi_h^0\|_{L^q}) \\ &\leq C_{p,q} (\|\mathbf{P}_h \phi - \mathbf{R}_h \phi\|_{L^p((0,T);L^q)} + \|\mathbf{P}_h \phi^0 - \phi_h^0\|_{L^q}), \end{aligned}$$

which proves (2.4).

The proof of Theorem 2.1 is completed.  $\blacksquare$

## 4.2 Proof of Corollary 2.1

(2.9) is a simple consequence of (2.1)-(2.4).

To prove (2.10), we apply an inverse inequality with (2.4) to (4.5) and we obtain

$$\begin{aligned} \|\partial_t(\phi_h - \mathbf{P}_h \phi)\|_{L^p((0,\tau);L^\infty)} &\leq C h^{-1-d/q} \|\partial_t(\phi_h - \mathbf{P}_h \phi)\|_{L^p((0,\tau);W^{-1,q})} \\ &\leq C_{p,q} (\|\phi^0\|_{W^{r+1,q}} + \|\phi\|_{L^p((0,T);W^{r+1,q})}) h^{r-1-d/q}, \end{aligned}$$

$$\|\phi_h - \mathbf{P}_h \phi\|_{L^p((0,\tau);L^\infty)} \leq C h^{-d/q} \|\phi_h - \mathbf{P}_h \phi\|_{L^p((0,\tau);L^q)}$$

$$\leq C_{p,q}(\|\phi^0\|_{W^{r+1,q}} + \|\phi\|_{L^p((0,T);W^{r+1,q})})h^{r+1-d/q}.$$

Therefore,

$$\begin{aligned} & \|\phi_h - \mathbf{P}_h \phi\|_{L^\infty((0,\tau);L^\infty)} \\ & \leq C\|\phi_h(0) - \mathbf{P}_h \phi(0)\|_{L^\infty} + C\|\phi_h - \mathbf{P}_h \phi\|_{L^p((0,\tau);L^\infty)}^{1-1/p} \|\partial_t(\phi_h - \mathbf{P}_h v)\|_{L^p((0,\tau);L^\infty)}^{1/p} \\ & \leq Ch^{-d/q}\|\phi_h(0) - \mathbf{P}_h \phi(0)\|_{L^q} + C\|\phi_h - \mathbf{P}_h \phi\|_{L^p((0,\tau);L^\infty)}^{1-1/p} \|\partial_t(\phi_h - \mathbf{P}_h v)\|_{L^p((0,\tau);L^\infty)}^{1/p} \\ & \leq C_{p,q}(\|\phi^0\|_{W^{r+1,q}} h^{r+1-d/q} + \|\phi\|_{L^p((0,T);W^{r+1,q})} h^{r+1-2/p-d/q}), \end{aligned}$$

where the last inequality holds when  $d/q < r+1$ .

Choosing  $p = q$ , we obtain

$$\|\mathbf{P}_h \phi - \phi_h\|_{L^\infty((0,T);L^\infty)} \leq C_{p,p}(\|\phi^0\|_{W^{r+1,\infty}} + \|\phi\|_{L^\infty((0,T);W^{r+1,\infty})})h^{r+1-(2+d)/p}. \quad (4.8)$$

Without loss of generality, we can assume that  $C_{p,p} \geq 2$  is an increasing function of  $p$  and define  $f(p) = p \ln C_{p,p}$ . Clearly,  $f$  is an increasing function of  $p$  and its inverse function exists. Moreover, choosing  $h_* = e^{f(d+2)/(d+2)}$ ,  $p = f^{-1}((d+2) \ln 1/h)$  and defining

$$\epsilon_h = (d+2)/f^{-1}\left((d+2) \ln \frac{1}{h}\right),$$

we have  $C_{p,p} = h^{-\epsilon_h}$ . When  $h < h_*$ , we have  $\epsilon_h \in (0, 1)$ ,  $\lim_{h \rightarrow 0} \epsilon_h = 0$  and  $C_{p,p} h^{r+1-(d+2)/p} = 2h^{r+1-\epsilon_h}$ , which imply that

$$\|\mathbf{P}_h \phi - \phi_h\|_{L^\infty((0,T);L^\infty)} \leq (\|\phi^0\|_{W^{r+1,\infty}} + \|\phi\|_{L^\infty((0,T);W^{r+1,\infty})})2h^{r+1-\epsilon_h}.$$

When  $h \geq h_*$ , we simply choose  $p = q = d+2$  in (4.8) and obtain

$$\|\mathbf{P}_h \phi - \phi_h\|_{L^\infty((0,T);L^\infty)} \leq (\|\phi^0\|_{W^{r+1,\infty}} + \|\phi\|_{L^\infty((0,T);W^{r+1,\infty})})Ch_*^{-1}h^{r+1}.$$

Combining the last two inequalities, we obtain

$$\|\mathbf{P}_h \phi - \phi_h\|_{L^\infty((0,T);L^\infty)} \leq (\|\phi^0\|_{W^{r+1,\infty}} + \|\phi\|_{L^\infty((0,T);W^{r+1,\infty})})Ch^{r+1-\min(1,\epsilon_h)}.$$

This completes the proof of (2.10). ■

## 5 Numerical examples

In this section, we present two numerical examples to support our theoretical analysis. All computations are performed by the software FreeFEM++.

*Example 5.1* We test the convergence rate of the Galerkin finite element solution for a parabolic equation with Lipschitz continuous coefficients, i.e., the equation

$$\begin{cases} \partial_t \phi - \nabla \cdot (A \nabla \phi) = f & \text{in } \Omega, \\ A \nabla \phi \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ \phi(x, 0) = \phi_0(x) & \text{for } x \in \Omega, \end{cases} \quad (5.1)$$

in the domain  $\Omega = (0, 1) \times (0, 1)$ , where

$$A = 3 + 0.1(x + y - t)^3 \sin\left(\frac{1}{(x + y - t)^2}\right),$$

$$f = e^t \sin(\pi x) \quad \text{and} \quad \phi_0 = \cos(\pi x) \cos(\pi y).$$

Clearly, the coefficient  $A$  is Lipschitz continuous and its second-order derivatives are unbounded. By the theory of parabolic equations, the exact solution of (5.1) satisfies that  $\phi \in L^p((0, T); W^{2,p})$  and  $\partial_t \phi \in L^p((0, T); L^p)$  for any  $1 < p < \infty$ . Under this regularity, (2.10) indicates that the numerical solution has an almost second-order convergence rate in the  $L^\infty$  norm.

We solve the above equation by the linear Galerkin FEM up to the time  $t = 1$ . A uniform triangulation is generated with  $M + 1$  nodes in each direction and a backward Euler scheme is used for the discretization in the time direction, where the time step  $\Delta t$  is chosen to be small enough compared with the mesh size  $h = 1/M$ . The numerical solution  $u_h$  is calculated with different mesh size  $h$ , and the difference between the numerical solutions at two consecutive meshes are presented Table 5.1, where the convergence rate  $O(h^\alpha)$  is calculated by the formula  $\alpha = \ln(|u_h - u_{h/2}|/|u_{h/2} - u_{h/4}|)/\ln 2$  at the finest two levels. We see that the convergence rate is about second order, which is consistent with our numerical analysis.

Table 5.1: Convergence rate of the numerical solution.

$h$	$\ u_h - u_{h/2}\ _{L^\infty}$
1/16	1.284E-03
1/32	3.411E-04
1/64	8.975E-05
convergence rate	$O(h^{1.9})$

*Example 5.2* We test the convergence rate of the scheme (2.11)-(2.12) for the equations of incompressible miscible flow in porous media. For this purpose, we consider the equations

$$\frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u}) \nabla c) + \mathbf{u} \cdot \nabla c = g, \quad (5.2)$$

$$-\nabla \cdot \left( \frac{1}{\mu(c)} \nabla p \right) = f, \quad (5.3)$$

in the circular domain  $\Omega = \{(x, y) : (x - 0.5)^2 + (y - 0.5)^2 < 0.5^2\}$ , subject to the boundary and initial conditions

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= \Psi, \quad D(\mathbf{u}) \nabla c \cdot \mathbf{n} = \Phi \quad \text{for } x \in \partial\Omega, \quad t \in [0, T], \\ c(x, 0) &= c_0(x) \quad \text{for } x \in \Omega, \end{aligned} \quad (5.4)$$

where

$$\mathbf{u} = -\frac{2}{\mu(c)} \nabla p, \quad D(\mathbf{u}) = 1 + 0.1|\mathbf{u}| \quad \text{and} \quad \mu(c) = 1 + c.$$

The functions  $f$ ,  $g$ ,  $\Psi$ ,  $\Phi$  and  $c_0$  are chosen corresponding to the exact solution

$$p = 100(x - t)^2 e^{-t}, \quad c = 0.5 + 0.2e^{-t} \cos x \sin y. \quad (5.5)$$



A quasi-uniform triangulation is generated by the software with  $M$  nodes of uniform distribution on the boundary  $\partial\Omega$ , and we solve the system (5.2)-(5.3) by the linear Galerkin FEM (2.11)-(2.12) up to the time  $t = 1$ . A linearized semi-implicit Crank–Nicolson scheme is used for the time discretization with an extremely small time step  $\Delta t = 2^{-14}$ . The numerical solution is then compared with the exact solution, and the  $L^\infty$  errors of the numerical solution are presented in Table 5.2 for different  $M$ , where the convergence rate  $O(h^\alpha)$  is calculated by the formulae  $\alpha = \ln(|\mathbf{u}_h - \mathbf{u}|/|\mathbf{u}_{h/2} - \mathbf{u}|)/\ln 2$  and  $\alpha = \ln(|c_h - c|/|c_{h/2} - c|)/\ln 2$ , respectively, at the finest mesh level. We can see that the convergence rate of the numerical solution is about second order.

Table 5.2:  $L^\infty$  errors of the numerical solution.

$M$	$\ \mathbf{u}_h - \mathbf{u}\ _{L^\infty}$	$\ c_h - c\ _{L^\infty}$
16	6.950E-02	7.594E-02
32	1.734E-02	1.720E-02
64	4.033E-03	3.823E-03
convergence rate	$O(h^{2.1})$	$O(h^{2.1})$

## 6 Conclusions

We have established an optimal  $L^p$ -norm and an almost optimal  $L^\infty$ -norm error estimate of Galerkin FEMs for the incompressible miscible flow in porous media with the commonly-used Bear-Scheidegger diffusion-dispersion model. Clearly, such a diffusion-dispersion tensor is only Lipschitz continuous and therefore, the traditional approach based on the classical elliptic projection may not be applicable. The analysis presented in this paper is based on a parabolic projection with Lipschitz continuous diffusion-dispersion tensors. The  $L^\infty$ -norm error estimate obtained is in the order of  $O(h^{r+1-\epsilon_h})$ , which is almost optimal since  $\epsilon_h \rightarrow 0$  as  $h \rightarrow 0$ . However, we do not know whether the optimal order  $h^{-\epsilon_h} = O(|\ln h|)$  holds for the nonlinear equations. In addition, our analysis only focuses on the semi-discrete schemes (spatial discretization). The stability and maximal regularity estimates of fully discrete finite element approximations for parabolic equations have not been investigated.

## Appendix: Proof of Lemma 3.1

We shall prove the lemma by applying the Leray–Schauder fixed point theorem [20].

**Lemma A.1 (Leray–Schauder)** *Let  $S_h^r$  be equipped with the maximum norm  $\|\cdot\|_{C(\overline{\Omega})}$ . Let  $\mathcal{A} : C([0, T]; S_h^r) \times [0, 1] \rightarrow C([0, T]; S_h^r)$  be a compact and continuous map such that the set*

$$\bigcup_{s \in [0, 1]} \{w \in C([0, T]; S_h^r) : \mathcal{A}(w, s) = w\}$$

*is bounded in  $C(\overline{\Omega}_T)$ , and  $\mathcal{A}(\cdot, 0) \equiv 0$ . Then there exists a fixed point  $w \in C([0, T]; S_h^r)$  satisfying  $\mathcal{A}(w, 1) = w$ .*

For any given  $c_h^0 \in C([0, T]; S_h^r)$  and  $s \in [0, 1]$ , we define  $\{P_h(t) \in S_h^{r+1}\}_{t \in [0, T]}$  and  $\{c_h(t) \in S_h^r\}_{t \in (0, T]}$  to be the solution of the following linear equations

$$\left( \frac{k(x)}{\mu(c_h^0)} \nabla P_h, \nabla \varphi_h \right) = \left( s(q_i - q_p), \varphi_h \right), \quad \forall \varphi_h \in S_h^{r+1}, \quad (\text{A.1})$$

$$\left( \Phi \partial_t c_h, \phi_h \right) + \left( D(\mathbf{u}_h) \nabla c_h, \nabla \phi_h \right) + \left( \mathbf{u}_h \cdot \nabla c_h, \phi_h \right) = s \left( \hat{c} q_i - c_h q_p, \phi_h \right), \quad \forall \phi_h \in S_h^r, \quad (\text{A.2})$$

where

$$\mathbf{u}_h = -\frac{k(x)}{\mu(c_h^0)} \nabla P_h,$$

with the initial condition  $c_h(0) = sc(0)$ . We denote by  $\mathcal{M}$  the map from  $(c_h^0, s)$  to  $P_h$  and by  $\mathcal{A}$  the map from  $(c_h^0, s)$  to  $c_h$ .

**Lemma A.2** *The map  $\mathcal{A} : C([0, T]; S_h^r) \times [0, 1] \rightarrow C([0, T]; S_h^r)$  is continuous and compact.*

*Proof* First, easy to check that for any given  $c_h^0$ , we have  $c_h = 0$  when  $s = 0$ .

Secondly, let  $P_h = \mathcal{M}(c_h^0, s)$ ,  $\bar{P}_h = \mathcal{M}(\bar{c}_h^0, \bar{s})$ ,  $c_h = \mathcal{A}(c_h^0, s)$  and  $\bar{c}_h = \mathcal{A}(\bar{c}_h^0, \bar{s})$ , and assume that  $c_h^0$  and  $\bar{c}_h^0$  are bounded in  $C(\bar{\Omega}_T)$ . Substituting  $\varphi_h = P_h$  into (A.1), we obtain  $\|P_h\|_{L^\infty((0, T); H^1)} \leq C$ , and similarly we also get  $\|\bar{P}_h\|_{L^\infty((0, T); H^1)} \leq C$ , which together with an inverse inequality imply that

$$\|P_h\|_{L^\infty((0, T); W^{1, \infty})} + \|\bar{P}_h\|_{L^\infty((0, T); W^{1, \infty})} \leq C_h, \quad (\text{A.3})$$

$$\|\mathbf{u}_h\|_{L^\infty(\Omega_T)} + \|\bar{\mathbf{u}}_h\|_{L^\infty(\Omega_T)} \leq C_h. \quad (\text{A.4})$$

Since

$$\left( \frac{k(x)}{\mu(\bar{c}_h^0)} \nabla (P_h - \bar{P}_h), \nabla \varphi_h \right) = - \left( \left( \frac{k(x)}{\mu(c_h^0)} - \frac{k(x)}{\mu(\bar{c}_h^0)} \right) \nabla P_h, \nabla \varphi_h \right) + \left( (s - \bar{s})(q_i - q_p), \varphi_h \right) \quad (\text{A.5})$$

for  $\varphi_h \in S_h^{r+1}$ , by substituting  $\varphi_h = P_h - \bar{P}_h$  into the equation, we derive that

$$\|P_h - \bar{P}_h\|_{L^\infty((0, T); H^1)} \leq C_h (\|c_h^0 - \bar{c}_h^0\|_{L^\infty((0, T); L^2)} + |s - \bar{s}|), \quad (\text{A.6})$$

which with an inverse inequality further implies that,

$$\|P_h - \bar{P}_h\|_{L^\infty((0, T); W^{1, \infty})} \leq C_h (\|c_h^0 - \bar{c}_h^0\|_{L^\infty((0, T); L^2)} + |s - \bar{s}|). \quad (\text{A.7})$$

The above inequality also shows that

$$\|\mathbf{u}_h - \bar{\mathbf{u}}_h\|_{L^\infty((0, T); L^\infty)} \leq C_h (\|c_h^0 - \bar{c}_h^0\|_{L^\infty((0, T); L^2)} + |s - \bar{s}|). \quad (\text{A.8})$$

Substituting  $\phi_h = c_h$  into (A.2), we further get

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} \|c_h\|_{L^2}^2 \right) + \left( D(\mathbf{u}_h) \nabla c_h, \nabla c_h \right) &\leq C \|\mathbf{u}_h\|_{L^\infty} \|\nabla c_h\|_{L^2} \|c_h\|_{L^2} + \|\hat{c} q_i\|_{L^2} \|c_h\|_{L^2} \\ &\leq \epsilon \|\nabla c_h\|_{L^2}^2 + C(\epsilon^{-1} \|\mathbf{u}_h\|_{L^\infty}^2 + 1) \|c_h\|_{L^2}^2 + \|\hat{c} q_i\|_{L^2}^2. \end{aligned}$$

Using Gronwall's inequality, we derive that

$$\|c_h\|_{L^\infty((0, T); L^2)} \leq C_h, \quad (\text{A.9})$$

and by an inverse inequality,

$$\|c_h\|_{L^\infty((0,T);W^{1,\infty})} + \|\bar{c}_h\|_{L^\infty((0,T);W^{1,\infty})} \leq C_h. \quad (\text{A.10})$$

Since

$$\begin{aligned} & \left( \Phi \partial_t (c_h - \bar{c}_h), \phi_h \right) + \left( D(\mathbf{u}_h) \nabla (c_h - \bar{c}_h), \nabla \phi_h \right) + \left( \mathbf{u}_h \cdot \nabla (c_h - \bar{c}_h), \phi_h \right) \\ &= \left( -\bar{s}(c_h - \bar{c}_h) q_p, \phi_h \right) + \left( -(s - \bar{s})(\hat{c} q_i - c_h q_p), \phi_h \right) - \left( (D(\mathbf{u}_h) - D(\bar{\mathbf{u}}_h)) \nabla \bar{c}_h, \nabla \phi_h \right) \\ & \quad - \left( (\mathbf{u}_h - \bar{\mathbf{u}}_h) \cdot \nabla \bar{c}_h, \phi_h \right) \end{aligned} \quad (\text{A.11})$$

for  $\phi_h \in S_h^r$ , by substituting  $\phi_h = c_h - \bar{c}_h$  into the above equation, we obtain

$$\begin{aligned} & \frac{d}{dt} \left( \frac{1}{2} \|c_h - \bar{c}_h\|_{L^2}^2 \right) + \left( D(\mathbf{u}_h) \nabla (c_h - \bar{c}_h), \nabla (c_h - \bar{c}_h) \right) + \left( \mathbf{u}_h \cdot \nabla (c_h - \bar{c}_h), \phi_h \right) \\ & \leq C \|\mathbf{u}_h\|_{L^\infty} \|\nabla (c_h - \bar{c}_h)\|_{L^2} \|c_h - \bar{c}_h\|_{L^2} + \|D(\mathbf{u}_h) - D(\bar{\mathbf{u}}_h)\|_{L^2} \|\nabla \bar{c}_h\|_{L^\infty} \|\nabla (c_h - \bar{c}_h)\|_{L^2} \\ & \quad + \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|_{L^2} \|\nabla \bar{c}_h\|_{L^\infty} \|c_h - \bar{c}_h\|_{L^2} + C_h |s - \bar{s}| \|c_h - \bar{c}_h\|_{L^2} \\ & \leq \epsilon \|\nabla (c_h - \bar{c}_h)\|_{L^2}^2 + C_h \epsilon^{-1} (\|c_h - \bar{c}_h\|_{L^2}^2 + \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|_{L^2}^2 + |s - \bar{s}|^2) \\ & \leq \epsilon \|\nabla (c_h - \bar{c}_h)\|_{L^2}^2 + C_h \epsilon^{-1} (\|c_h - \bar{c}_h\|_{L^2}^2 + \|c_h^0 - \bar{c}_h^0\|_{L^2}^2 + |s - \bar{s}|^2). \end{aligned}$$

By Gronwall's inequality again, we derive that

$$\|c_h - \bar{c}_h\|_{L^\infty((0,T);L^2)} \leq C_h (\|c_h^0 - \bar{c}_h^0\|_{L^2} + |s - \bar{s}|^2).$$

which in turn produces

$$\|c_h - \bar{c}_h\|_{L^\infty((0,T);W^{1,\infty})} \leq C_h (\|c_h^0 - \bar{c}_h^0\|_{L^2} + |s - \bar{s}|). \quad (\text{A.12})$$

From (A.11) we see that

$$\begin{aligned} \left| \left( \Phi \partial_t (c_h - \bar{c}_h), \phi_h \right) \right| & \leq C \left( \|D(\mathbf{u}_h)\|_{L^\infty} \|\nabla (c_h - \bar{c}_h)\|_{L^2} + \|\mathbf{u}_h\|_{L^\infty} \|\nabla (c_h - \bar{c}_h)\|_{L^2} \right) \|\phi_h\|_{H^1} \\ & \quad + C \left( \|(c_h - \bar{c}_h) q_p\|_{L^2} + \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|_{L^\infty} \|\nabla \bar{c}_h\|_{L^2} \right) \|\phi_h\|_{H^1} \\ & \leq C_h (\|c_h^0 - \bar{c}_h^0\|_{L^2} + |s - \bar{s}|) \|\phi_h\|_{H^1} \\ & \leq C_h (\|c_h^0 - \bar{c}_h^0\|_{L^2} + |s - \bar{s}|) \|\phi_h\|_{L^2}, \end{aligned}$$

which leads to

$$\|\partial_t (c_h - \bar{c}_h)\|_{L^\infty((0,T);L^2)} \leq C_h (\|c_h^0 - \bar{c}_h^0\|_{L^2} + |s - \bar{s}|).$$

With an inverse inequality, we further derive that

$$\|\partial_t (c_h - \bar{c}_h)\|_{L^\infty((0,T);L^\infty)} \leq C_h (\|c_h^0 - \bar{c}_h^0\|_{L^\infty((0,T);L^2)} + |s - \bar{s}|). \quad (\text{A.13})$$

By applying the inverse inequality to (A.13), we also derive that (with  $\bar{s} = 0$  and  $\bar{c}_h = 0$ )

$$\|\partial_t \nabla c_h\|_{L^\infty((0,T);L^\infty)} \leq C_h. \quad (\text{A.14})$$

The inequalities (A.12) and (A.13) imply that

$$\|c_h - \bar{c}_h\|_{W^{1,\infty}(\Omega_T)} \leq C_h(\|c_h^0 - \bar{c}_h^0\|_{C(\bar{\Omega}_T)} + |s - \bar{s}|). \quad (\text{A.15})$$

Since  $W^{1,\infty}(\Omega_T)$  is compactly embedded into  $C(\bar{\Omega}_T)$ , we derive that the map from  $(c_h^0, s)$  to  $c_h$  is compact and continuous. ■

We proceed to prove Lemma 3.1. From (A.10), we see that the set

$$\bigcup_{s \in [0,1]} \{w \in C([0, T]; S_h^r) : A(w, s) = w\}$$

is bounded in  $C(\bar{\Omega}_T)$ . By Lemma A.1, there exists a  $c_h \in C([0, T]; S_h^r)$  and  $P_h = \mathcal{M}(c_h, 1) \in C([0, T]; S_h^r)$  as the solution of (2.11)-(2.12). From (A.3) and (A.10) we know that  $P_h \in L^\infty((0, T); W^{1,\infty})$  and  $c_h \in W^{1,\infty}(\bar{\Omega}_T)$ , and from (A.14) we derive (3.1).

Uniqueness of the solution can be proved easily and the proof of Lemma 3.1 is completed. ■

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